

A REMARK ON THE UENO-CAMPANA'S THREEFOLD

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Dedicated to Fabrizio Catanese on his 65th birthday

ABSTRACT. We show that the Ueno-Campana's threefold cannot be obtained as the blow-up of any smooth threefold along a smooth centre, answering negatively a question raised by Oguiso and Truong.

1. INTRODUCTION

Let $E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ be the complex elliptic curve of period τ . There exist exactly two elliptic curves with automorphism group bigger than $\{\pm 1\}$: these are defined respectively by the periods $\sqrt{-1}$ and the cubic root of unity $\omega := (-1 + \sqrt{-3})/2$.

We consider the diagonal action of the cyclic group generated by $\sqrt{-1}$ (resp. $-\omega$) on the product

$$E_{\sqrt{-1}} \times E_{\sqrt{-1}} \times E_{\sqrt{-1}} \quad (\text{resp. } E_\omega \times E_\omega \times E_\omega)$$

and we denote by X_4 (resp. X_6) the minimal resolution of their quotients:

$$E_{\sqrt{-1}} \times E_{\sqrt{-1}} \times E_{\sqrt{-1}} / \langle \sqrt{-1} \rangle \quad (\text{resp. } E_\omega \times E_\omega \times E_\omega / \langle -\omega \rangle).$$

The minimal resolutions are obtained by a single blow-up at the maximal ideal of each singular point of the quotients above.

The threefolds X_4 and X_6 have been extensively studied in the past. In particular, they admit an automorphism of positive entropy (e.g. see [Ogu15] for more details). The variety X_4 is now referred as the

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Ueno-Campana's threefold. It has been recently shown that X_4 and X_6 are rational. Indeed, Oguiso and Truong [OT15] showed the rationality of X_6 , and Colliot-Thélène [CT15] showed the rationality of X_4 , after the work of Catanese, Oguiso and Truong [COT14]. The unirationality of X_4 was conjectured by Ueno [Uen75], whilst Campana asked about the rationality of X_4 in [Cam11].

The aim of this note is to give a negative answer to the following question raised by Oguiso and Truong (see [Ogu15][Question 5.11] and [Tru15][Question 2]).

Question 1.1. *Can X_4 or X_6 be obtained as the blow-up of \mathbb{P}^3 , $\mathbb{P}^2 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along smooth centres?*

Our main result is the following:

Theorem 1.2. *Let A be an abelian variety of dimension three and let G be a finite group acting on A such that the quotient map*

$$\rho: A \rightarrow Z = A/G$$

is étale in codimension 2.

Assume that there exists a resolution $f: X \rightarrow Z$ given by the blow-up of the singular points of Z and such that the exceptional divisor at each singular point of Z is irreducible.

Then X cannot be obtained as the blow-up of a smooth threefold along a smooth centre.

Note that Theorem 1.2 provides a negative answer to Question 1.1. Very recently, Lesieutre [Les15] announced that Question 1.1 admits a negative answer, using different methods.

2. PRELIMINARY RESULTS

We use some of the methods introduced in [CT14]. Let X be a normal projective threefold with isolated quotient singularities. Given a basis $\gamma_1, \dots, \gamma_m$ of $H^2(X, \mathbb{C})$, the *cubic form associated to X* is the homogeneous polynomial of degree 3 defined by:

$$F_X(x_1, \dots, x_m) = (x_1\gamma_1 + \dots + x_m\gamma_m)^3 \in \mathbb{C}[x_1, \dots, x_m].$$

Note that, modulo the natural action of $\mathrm{GL}(m, \mathbb{C})$, the cubic F_X does not depend on the choice of the base and it is a topological invariant of the underlying manifold X (see [OVdV95] for more details). In particular, if

$$\mathcal{H}_{F_X} = (\partial_{x_i} \partial_{x_j} F_X)_{i,j=1,\dots,m}$$

denotes the *Hessian matrix* associated to F_X and $p \in H^2(X, \mathbb{C})$, then the rank of \mathcal{H}_{F_X} at p is well-defined.

The following basic tool was used in [CT14] in a more general context. We provide a proof for the reader's convenience.

Lemma 2.1. *Let Y be a normal projective threefold with isolated quotient singularities and let $f: X \rightarrow Y$ be the blow-up of Y along a point $q \in Y$ (resp. a curve $C \subseteq Y$). Assume that the exceptional divisor of f is irreducible and let E be its class in $H^2(X, \mathbb{C})$.*

Then the rank of the Hessian matrix \mathcal{H}_{F_X} of F_X at E is one (resp. at most two).

Note that by [CT14][Lemma 2.7 and Lemma 2.12] the rank of \mathcal{H}_{F_X} is never zero.

Proof. We have $H^2(X, \mathbb{C}) = \langle E, f^*(\gamma_1), \dots, f^*(\gamma_m) \rangle$ where $\gamma_1, \dots, \gamma_m$ is a basis of $H^2(Y, \mathbb{C})$.

Consider the cubic form F_X associated to X with respect to this basis:

$$F_X(x_0, \dots, x_m) = (x_0 E + \sum_{i=1}^m x_i f^*(\gamma_i))^3.$$

Since $f^*(\gamma_i) \cdot f^*(\gamma_j) \cdot E = 0$ for all $i, j = 1, \dots, m$, we have

$$F_X(x_0, \dots, x_m) = x_0^3 E^3 + 3 \sum_{i=1}^m x_0^2 x_i E^2 f^*(\gamma_i) + \left(\sum_{i=1}^m x_i f^*(\gamma_i) \right)^3.$$

Let $a = E^3$ and let $b_i = E^2 f^*(\gamma_i)$ for $i = 1, \dots, m$. Note that if f is the blow-up of a point $q \in Y$ then $b_1 = \dots = b_m = 0$.

Thus, we have

$$F_X(x_0, \dots, x_m) = ax_0^3 + 3 \sum_{i=1}^m b_i x_0^2 x_i + G(x_1, \dots, x_m),$$

where G is a homogeneous cubic polynomial in the variables x_1, \dots, x_m , i.e. it does not depend on x_0 . Let $p = y_0 E + \sum_{i=1}^m y_i f^*(\gamma_i) \in H^2(X, \mathbb{C})$, for some $y_0, \dots, y_m \in \mathbb{C}$ and let $p' = (y_1, \dots, y_m)$. After removing the first row and the first column, the Hessian matrix $\mathcal{H}_{F_X}(p)$ of F_X at p , coincides with the Hessian matrix $\mathcal{H}_G(p')$ of G at p' .

In particular, if $p = E$, then $p' = (0, \dots, 0)$ and $\mathcal{H}_G(p')$ is the zero matrix. Thus, the rank of the Hessian of F_X at p is at most two. In addition, if $b_1 = \dots = b_m = 0$, then the rank of \mathcal{H}_F at p is exactly one. \square

3. PROOFS

Lemma 3.1. *Let A be an abelian variety of dimension 3 and let G be a finite group acting on A such that the quotient map $\rho: A \rightarrow Z = A/G$*

is étale in codimension 2. Let F_Z be the cubic form associated to Z and let $p \in H^2(Z, \mathbb{C})$ such that $\text{rk } \mathcal{H}_{F_Z}(p) \leq 1$.

Then $p = 0$.

Proof. The morphism ρ induces an immersion of vector spaces

$$\rho^*: H^2(Z, \mathbb{C}) \rightarrow H^2(A, \mathbb{C}).$$

Thus, there exists a basis of $H^2(A, \mathbb{C})$ such that if F_A is the cubic associated to A with respect to this basis and d is the degree of ρ , then

$$F_Z(x_1, \dots, x_m) = d \cdot F_A(x_1, \dots, x_m, 0, \dots, 0).$$

It is enough to show that if $q \in H^2(A, \mathbb{C})$ is such that the rank of \mathcal{H}_{F_A} at q is not greater than one, then $q = 0$.

Write $A = \mathbb{C}^3/\Gamma$ and consider z_1, z_2, z_3 coordinates on \mathbb{C}^3 . Then a basis of $H^2(A, \mathbb{C})$ is given by

$$\begin{aligned} z_{ij} &= dz_i \wedge dz_j & 1 \leq i < j \leq 3, \\ z_{i\bar{j}} &= dz_i \wedge d\bar{z}_j & i, j \in \{1, 2, 3\}, \\ z_{\bar{i}\bar{j}} &= d\bar{z}_i \wedge d\bar{z}_j & 1 \leq i < j \leq 3. \end{aligned}$$

For any $x \in H^2(A, \mathbb{C})$, let $x_{ij}, x_{i\bar{j}}$ and $x_{\bar{i}\bar{j}}$ be the coordinates of x with respect to the basis above and let F'_A be the cubic associated to this basis. It is enough to show that if $q \in H^2(A, \mathbb{C})$ is such that the rank of $\mathcal{H}_{F'_A}$ at q is not greater than one, then $q = 0$. Let $q_{ij}, q_{i\bar{j}}$ and $q_{\bar{i}\bar{j}}$ be the coordinates of q .

The (2×2) -minor of $\mathcal{H}_{F'_A}$ at x defined by the rows corresponding to x_{12} and x_{13} and the columns corresponding to $x_{2\bar{1}}$ and $x_{3\bar{1}}$ is given by

$$\begin{pmatrix} 0 & 6x_{2\bar{3}} \\ 6x_{2\bar{3}} & 0 \end{pmatrix}.$$

It follows that $q_{2\bar{3}} = 0$. By choosing suitable (2×2) -minors, it follows easily that each coordinate of q is zero. Thus, the claim follows. \square

Proof of Theorem 1.2. Suppose not. Then there exists a smooth projective threefold Y such that X can be obtained as the blow-up $g: X \rightarrow Y$ at a smooth centre. Let E be the exceptional divisor of g . Let k be the number of singular points of Z and let E_1, \dots, E_k be the exceptional divisors on X corresponding to the singular points of Z .

We want to prove that $E = E_i$ for some $i = 1, \dots, k$. Denote by p the class of E in $H^2(X, \mathbb{C})$. Lemma 2.1 implies that the rank of \mathcal{H}_{F_X} at p is not greater than two.

Let $\gamma_1, \dots, \gamma_m \in H^2(Z, \mathbb{C})$ be a basis and let F_Z be the associated cubic form. Then $f^*\gamma_1, \dots, f^*\gamma_m, [E_1], \dots, [E_k]$ is a basis of $H^2(X, \mathbb{C})$ and if F_X denotes the associated cubic form, we have

$$F_X(x_1, \dots, x_m, y_1, \dots, y_k) = F_Z(x_1, \dots, x_m) + \sum_{i=1}^k a_i y_i^3,$$

where $a_i = E_i^3$ is a non-zero integer, for $i = 1, \dots, k$.

Thus, the Hessian matrix of F_X is composed by two blocks: one is the Hessian matrix of F_Z and the other one is a diagonal matrix, whose only non-zero entries are $6a_i$ for $i = 1, \dots, k$. We may write $p = (p^0, p^1) = (p_1^0, \dots, p_m^0, p_1^1, \dots, p_k^1)$. We have $\text{rk } \mathcal{H}_{F_Z}(p^0) \leq 2$.

We distinguish two cases. If $\text{rk } \mathcal{H}_{F_Z}(p^0) = 2$, then $p^1 = (0, \dots, 0)$ and in particular E is numerically equivalent to f^*D , for some pseudo-effective Cartier divisor D on Z . Since A is abelian, it follows that ρ^*D is a nef divisor. Thus E is nef, a contradiction.

If $\text{rk } \mathcal{H}_{F_Z}(p^0) \leq 1$, then Lemma 3.1 implies that $p^0 = 0$. Thus,

$$E \equiv c_s E_s + c_t E_t$$

for some distinct $s, t \in \{1, \dots, k\}$ and c_s, c_t rational numbers. Since E is effective non-trivial, at least one of the c_i is positive. By symmetry, we may assume $c_s > 0$. By the negativity lemma, the divisor E_s is covered by rational curves C such that $E_s \cdot C < 0$. Since E_s and E_t are disjoint, it follows that $E \cdot C < 0$, which implies that C is contained in E . Thus E_s is contained in E . Since E is prime, it follows that $E = E_s$ and $c_t = 0$.

Finally, note that g contracts $E = E_s$ to a point, as otherwise there exists a small contraction $\eta: Y \rightarrow Z$ and in particular Z is not \mathbb{Q} -factorial, a contradiction. Thus, $g: X \rightarrow Y$ is the contraction of E_s to the corresponding singular point on Z , which is again a contradiction. The claim follows. \square

Remark 3.2. As K. Oguiso kindly pointed out to us, the same proof shows that if $f: X \rightarrow Z$ is as in Theorem 1.2 and g is an automorphism on X then the set of exceptional divisors of f is invariant with respect to g . Thus, there exists a positive integer m such that the power g^m descends to an automorphism on Z .

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